User Localization in MmWave Cells: A Non-Adaptive Quantitative Group Testing Approach based on Sparse Graph Codes

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Problem Setup

Quantitative group testing (QGT): Result of a test is the number of defective items in the tested group.

Problem: Identify all defective items in a given population of N items for the following settings:

- Deterministic Model: There are exactly K defective items for a given integer $1 \leq K \leq N$
- Randomized Model: Each item is defective with probability $\frac{K}{N}$, independently from other items, for a given integer $1 \leq K \leq N$.

Basic Notation:

- $\mathbf{x} \in \{0,1\}^N$: Non-zero values correspond to the defective items, and zero values correspond to the non-defective items
- $\mathbf{A} \in \{0,1\}^{m \times N}$: The measurement matrix
- $\mathbf{y} = \mathbf{Ax} \in \{\mathbb{Z}_{\geq 0}\}^m$: Test results vector

$$\mathbf{Ax} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Goal

Design a testing matrix $\mathbf{A}$ that has a small number of rows (tests), m, and can identify the defective items given the test results $\mathbf{y}$.

Connecting QGT & Coding Theory

t-separable matrix: A binary matrix is t-separable over a field $\mathbb{F}$ if the sum of any set of t columns (over $\mathbb{F}$) is distinct.

- If a matrix with n columns is t-separable (for some $t \leq n$) over a field $\mathbb{F}$, then it is also i-separable over $\mathbb{F}$ for any $1 \leq i \leq t$.

Idea: Any t-separable matrix over any field $\mathbb{F}$ can be used as a measurement matrix for identifying t or fewer defective items in the QGT problem.

Challenge: The construction of an optimal t-separable matrix (with minimum number of rows) for an arbitrary t is an open problem.

A Near-Optimal Solution: Using a parity-check matrix of a binary t-error correcting BCH code.

Proposed Algorithm: Encoding

Let $G \triangleq G_{\ell,r}(N,M)$ be a randomly chosen bi-regular (left-and-right regular) bipartite graph:

- N and M: number of the left and right nodes
- ℓ and r: degree of the left and right nodes

Let $\mathbf{T}_G = [\mathbf{t}_1^{\mathsf{T}}, \dots, \mathbf{t}_M^{\mathsf{T}}]^{\mathsf{T}} \in \{0,1\}^{M \times N}$ be the adjacency matrix of G;

Let $\mathbf{H} = [\mathbf{h}_1^{\mathsf{T}}, \dots, \mathbf{h}_{t\log_2(r+1)}^{\mathsf{T}}]^{\mathsf{T}} \in \{0,1\}^{(t\log_2(r+1)) \times r}$ be a parity-check matrix of a binary t-error correcting $(r, r - t\log_2(r+1))$ BCH code;

Construct the signature matrix $\mathbf{U} \in \{0,1\}^{(t\log_2(r+1)+1) \times r}$ and the measurement matrix $\mathbf{A} \in \{0,1\}^{M(t\log_2(r+1)+1) \times N}$ as follows:

Let $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_r]$ where $\mathbf{u}_i = [1, \mathbf{h}_i^{\mathsf{T}}]^{\mathsf{T}}$, and let $\mathbf{A} = [\mathbf{A}_1^{\mathsf{T}}, \cdots, \mathbf{A}_M^{\mathsf{T}}]^{\mathsf{T}}$, where

$$\mathbf{A}_i = [\mathbf{0}, \dots, \mathbf{0}, \mathbf{u}_1, \mathbf{0}, \dots, \mathbf{u}_2, \mathbf{0}, \dots, \mathbf{u}_r]$$

when $\mathbf{t}_i = [0, \dots, 0, 1, 0, \dots, 1, 0, \dots, 1]$.

Proposed Algorithm: Decoding

A t-*Resolvable* Right Node: A right node that is connected to t or fewer defective items.

Resolving a t-Resolvable Right Node:

- The observation vector corresponding to the i-th right node:

$$\mathbf{z}_i = [z_{i,1}, z_{i,2}, \cdots, z_{i,t\log(r+1)+1}]^{\mathsf{T}} = \mathbf{A}_i\mathbf{x}$$

- Let $\mathbf{z}_i = [\mathbf{z}_i^{(1)^{\mathsf{T}}}, \mathbf{z}_i^{(2)^{\mathsf{T}}}]^{\mathsf{T}}$, where $\mathbf{z}_i^{(1)} = z_{i,1}$ and $\mathbf{z}_i^{(2)} = [z_{i,2}, \cdots, z_{i,t\log(r+1)+1}]^{\mathsf{T}}$:

$$\begin{bmatrix} \mathbf{z}_i^{(1)} \\ \mathbf{z}_i^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 1 & 0 & \dots & 1 \\ 0 & \dots & 0 & \mathbf{h}_1 & 0 & \dots & \mathbf{h}_2 & 0 & \dots & \mathbf{h}_r \end{bmatrix} \mathbf{x}$$

- $\mathbf{z}_i^{(1)}$ is used to find the number of defective items (j) connected to the i-th right node
- $\mathbf{z}_i^{(2)}$ (under modulo 2) is used by the BCH decoder to locate the j errors (for $0 \leq j \leq t$)

Iterative Peeling Decoding: In each iteration,

(i) Resolve all the t-*resolvable* right nodes

(ii) Peel the edges connected to the recovered defective items off the graph

(iii) Terminate if $\nexists$ any t-resolvable right node

Example

Let $N = 14$ and $M = 4$, and let $G = G_{2,7}(14,4)$ be a bi-regular bipartite graph ($\ell = 2$ and $r = 7$) where

$$\mathbf{T}_G = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

First, we construct the matrices H and U by using a binary $t = 1$-error correcting $(7,4)$ BCH code:

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Next, we construct the measurement matrix A as:

$$\mathbf{A} = \left[\begin{array}{cccccccccccccc} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Suppose the number of defective items is $K = 3$, and the items 1, 4, and 10 are defective, i.e., $\mathbf{x} = [1,0,0,1,0,0,0,0,0,1,0,0,0,0]^T$. Then,

$$\mathbf{y} = \mathbf{Ax} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_5 \\ \mathbf{u}_2 + \mathbf{u}_5 \\ \mathbf{u}_1 + \mathbf{u}_2 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{z}_1^{\mathsf{T}} \\ \mathbf{z}_2^{\mathsf{T}} \\ \mathbf{z}_3^{\mathsf{T}} \\ \mathbf{z}_4^{\mathsf{T}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix}$$

- The 1st and 2nd right nodes are 1-resolvable, and using a BCH decoder, we find that the items 1 and 10 are defective.
- Removing the contributions of the items 1 and 10, the 3rd and 4th right nodes become 1-resolvable, and using a BCH decoder, we find that the item 4 is defective.

Main Theorem

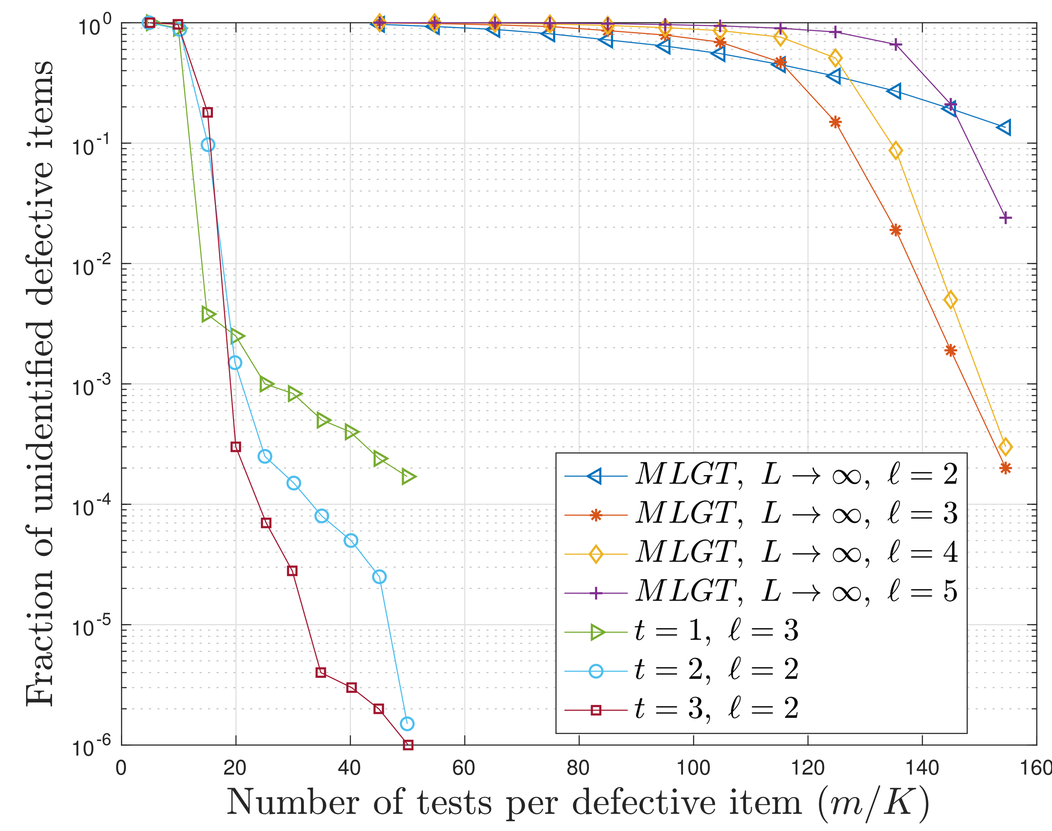
For both the deterministic and randomized models of the defective items, the proposed algorithm recovers all the defective items with probability approaching one (as N and K grow unbounded) with at most $m = c(t)K(t\log(\frac{N\ell}{c(t)K}+1)+1)+1$ tests, where $c(t)$ is a constant that depends only on t.

The following table lists the constant $c(t)$ and the optimal left degree ℓ^* for $t \in \{1,2,\cdots,6\}$:

t	1	2	3	4	5	6
$c(t)$	1.222	0.597	0.388	0.294	0.239	0.202
ℓ^*	3	2	2	2	2	2

Comparison Results

- The results of our theoretical analysis show that when $t \in \{1,2,3\}$ the required number of tests is less than that for larger values of t.



- Our simulation results also confirm that the proposed algorithm (for $t \in \{1,2,3\}$) significantly outperforms the Multi-Level Group Testing (MLGT) scheme of [2], which is one of the best existing schemes for QGT.

References

[1] E. Karimi, F. Kazemi, A. Heidarzadeh, and A. Sprintson "A Simple and Efficient Strategy for the Coin Weighing Problem with a Spring Scale," *IEEE International Symposium on Information Theory (ISIT'18)*, 2018.

[2] P. Abdalla, A. Reisizadeh, R. Pedarsani, "Multilevel group testing via sparse-graph codes," *51st Asilomar Conference on Signals, Systems, and Computers*, 2017.

[3] E. Karimi, F. Kazemi, A. Heidarzadeh, K. R. Narayanan and A. Sprintson "Sparse Graph Codes for Non-adaptive Quantitative Group Testing," *Submitted to IEEE International Symposium on Information Theory (ISIT'19)*, 2019.